Economics of Politics, Answers to Problem set #2
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Exercise 1.

First of all, let us describe party i's payoff function. Here parties are policy-motivated. They are concerned not about ego rent but about policies implemented. Suppose that party L promises p_L and party R does p_R in the election. If party L win the election with probability q, then its expected payoff, $E\pi_L$, is expressed as

$$E\pi_L = q_L U_L(p_L) + (1 - q_L)U_L(p_R),$$

since policy p_L is implemented with probability q_L and policy p_R is with probability $1 - q_L$. Similarly, party R's expected payoff, $E\pi_R$, is

$$E\pi_L = q_L U_R(p_L) + (1 - q_L)U_R(p_R).$$

Note that I am assuming that the utility functions are continuous throughout this problem. Otherwise, the convergence result will not hold in general.

The probability that party L wins depends on if its campaign promise, p_L, is closer to the median voter's ideal policy p_m. That is,

$$q_L = \begin{cases} 1 & \text{if } |p_L - p_m| < |p_R - p_m| \\ 1/2 & \text{if } |p_L - p_m| = |p_R - p_m| \\ 0 & \text{if } |p_L - p_m| > |p_R - p_m|, \end{cases} \tag{1}$$

where $|x|$ denotes the absolute value of x.

(1) When $p_L = p_R = p_m$, we have $q_L = 1/2$, and the payoffs are

$$E\pi_L = U_L(p_m) \tag{2}$$

and

$$E\pi_R = U_R(p_m). \tag{3}$$

Suppose now that party L deviates and chooses $p_L \neq p_m$. Then, because the probability of party L's winning the election becomes zero, the payoff to party L does not change from (4). The same result applies to party R when it changes its promise from p_m. Accordingly, $p_L = p_R = p_m$ is a Nash equilibrium.

In fact, this is a unique Nash equilibrium of this game. In the next two questions you will see the uniqueness.

(2) If $p_L < p_R \leq p_m$, then $q_L = 0$ and $E\pi_L = U_L(p_R)$ and $E\pi_R = U_R(p_R)$. In such a situation, party R can increase its payoff by making p_R closer to its ideal $p = 1$ without changing the probability of winning, because $U_R(p)$ is single-peaked and maximized at $p = 1$.

This consideration clarifies that there is no equilibrium with either $p_L < p_R \leq p_m$ or $p_m \leq p_L < p_R$. Similar reasoning applies to the case of either $p_R < p_L \leq p_m$ or $p_m \leq p_R < p_L$, in which respective cases, party L and R have incentives to make promises closer to their ideals.

(3) Suppose $p_L < p_m < p_R$ and the median voter is indifferent between the two promises. Then, party L's expected payoff is

$$E\pi_L = 0.5U_L(p_L) + 0.5U_L(p_R).$$

In such a situation, party L can win the election for sure by making p_L a little closer to p_m than p_R (by the amount, say, $\Delta p > 0$). By doing so, its expected payoff changes to $U_L(p_L + \Delta p)$. Note that the difference between $U_L(p_L + \Delta p)$ and $U_L(p_L)$ is negligible when Δp is sufficiently close to zero. Accordingly, by slightly changing p_L toward p_m, party L improves its payoff approximately by the amount of $U_L(p_L) - U_L(p_R)$, and hence there is no Nash equilibrium with p_L and p_R being different from p_m.

Exercise 2.

(1) Because of the median voter theorem, party L's victory occurs if and only if its policy platform, p_L, is closer to the median voter's ideal, p_m, than party R's one, p_R. Because $p_L > p_R$ cannot happen in any Nash equilibrium, the probability of party L's victory is formulated as

$$q_L = \text{Prob}(p_m \leq (p_L + p_R)/2) = H\left(\frac{p_L + p_R}{2}\right).$$

Using this yields party L's payoff function as

$$E\pi_L = H\left(\frac{p_L + p_R}{2}\right)U_L(p_L) + \left\{1 - H\left(\frac{p_L + p_R}{2}\right)\right\}U_L(p_R). \tag{4}$$

Note that in this case, because the median voter's position follows a continuous distribution, the probability that a tie will occur is negligibly small and we can ignore it in the formulation of q_L.

(2) First of all, let us see that policy convergence is not an equilibrium outcome. Suppose that there occurs policy convergence such that $p_L = p_R = p^*$. Then, party L will win the election with probability $1/2$ and its expected payoff is $E\pi_L = U_L(p^*)$, since both parties choose the same policy when in office.

What will happen if party L deviates to $p_L < p^*$. In contrast to the model in the previous problem, party L still has a chance of winning the election. The probability is equal to

$$q_{L1} = H\left(\frac{p_L + p^*}{2}\right)$$

and party L's expected payoff is

$$E\pi_L = q_{L1}U_L(p_L) + \{1 - q_{L1}\}U_L(p^*),$$

which is strictly greater than $U_L(p^*)$ because $U_L(p_L) > U_L(p^*)$. Thus, there is no Nash equilibrium with policy convergence realized.

Now consider party L's best response to party R's arbitrary promise, p_R. Differentiating (4) yields

$$\frac{\partial E\pi_L}{\partial p_L} = \frac{1}{2}h\left(\frac{p_L + p_R}{2}\right)\{U_L(p_L) - U_L(p_R)\} + H\left(\frac{p_L + p_R}{2}\right)U_L'(p_L) = 0.$$

Therefore in a Nash equilibrium with p_L^* and p_R^* being promised, we must have

$$\frac{1}{2}h\left(\frac{p_L^* + p_R^*}{2}\right)\{U_L(p_L^*) - U_L(p_R^*)\} = -H\left(\frac{p_L^* + p_R^*}{2}\right)U_L'(p_L^*)$$

and

$$\frac{1}{2}h\left(\frac{p_L^* + p_R^*}{2}\right)\{U_R(p_R^*) - U_R(p_L^*)\} = \left\{1 - H\left(\frac{p_L^* + p_R^*}{2}\right)\right\}U_R'(p_R^*)$$

Because $U_L' < 0$ and $U_R' > 0$, these conditions demonstrate that $p_L^* < p_R^*$.

(3) When $H(\cdot)$ is uniform, the above conditions are reduced into

$$\frac{1}{2}\{U_L(p_L^*) - U_L(p_R^*)\} = -\left(\frac{p_L^* + p_R^*}{2}\right)U_L'(p_L^*)$$

and

$$\frac{1}{2}\{U_R(p_R^*) - U_R(p_L^*)\} = \left\{1 - \left(\frac{p_L^* + p_R^*}{2}\right)\right\}U_R'(p_R^*)$$

From here, unfortunately, we cannot solve p_L^* and p_R^* without specifying parties' utility functions, which I am very sorry that I forgot.

Let us assume that $U_L = -p_L^2$ and $U_R = -(p_R - 1)^2$, as are the same with voters whose ideal policies are $p = 0$ and $p = 1$, respectively. Using these specifications, we can rewrite the above conditions further into

$$-\frac{p_L^* - p_R^*}{2} = p_L^*$$

and

$$\frac{p_L^* - p_R^*}{2} = -(1 - p_R^*).$$

Solving these equations yields $p_L^* = 1/4$ and $p_R^* = 3/4$.

Exercise 3.

(1) Suppose that the voters will reelect the incumbent with probability $p(e)$ if he chooses effort e in period 1. Then, the incumbent's expected payoff at the timing of taking office in period 1 is $R - e^2 + p(e)\delta R$. To make the incumbent choose e, the payoff must be as large as R, since he has already taken office.

Consider that the voters coordinate their actions to take a retrospective voting strategy,

$$p(e) = \begin{cases} 1 & \text{if } e \geq \overline{e} \\ 0 & \text{otherwise.} \end{cases}$$

Then, provided that this strategy is credible to the incumbent, he will choose $\overline{e}$ if and only if

$$R - \overline{e}^2 + \delta R \geq R,$$

from which the optimal level of $\overline{e}$ for the voters is given by $\overline{e} = \sqrt{\delta R}$.

(2) Because even if the incumbent implements effort greater than $\overline{e}$, the voters will vote against him when the challenger is female.

(3) A credible retrospective voting strategy is as follows.

$$p(e) = \begin{cases} 1 & \text{if } e \geq \overline{e} \text{ and the challenger is male} \\ 0 & \text{otherwise.} \end{cases}$$

Given this voting strategy, the incumbent has an incentive to choose $\overline{e}$ if and only if

$$R - \overline{e}^2 + \delta R/2 \geq R, \tag{5}$$

since the probability of a male challenger appearing is 1/2. From this condition, the maximum level of effort that the voters can deduce from the incumbent is $\overline{e} = \sqrt{\delta R/2}$, which is lower than the level in question (1) due to reduced credibility.

(4) Suppose that the voters use the incumbent's campaign promise made in the period 1 election, say e_I, as the critical level with which to judge his reelection, $\overline{e}$, in the retrospective voting strategy that they each will take at the beginning of period 2. As shown in question (3), the incumbent has no incentive to keep campaign promises with $e_I > \sqrt{\delta R/2}$. On the other hand, any promises such that $e_I \leq \sqrt{\delta R/2}$ are credible. Accordingly, given e_B, the probability of candidate A's winning, q_A, is expressed as follows.

$$q_A = \begin{cases} 1 & \text{if } e_B < e_A \leq \sqrt{\delta R/2} \text{ or } e_A \leq \sqrt{\delta R/2} < e_B \\ 0 & \text{if } e_A < e_B \leq \sqrt{\delta R/2} \text{ or } e_B \leq \sqrt{\delta R/2} < e_A \\ 1/2 & \text{otherwise.} \end{cases}$$

(5) First of all, a Nash equilibrium occurs only with $q_L = 1/2$ as long as it exists, because in the other cases (5) holds with strict inequality for the winning candidate, which means that the losing candidate has an incentive to mimic his rival's campaign promise and make a tie at the worst. The cases consistent with $q_L = 1/2$ are of two types. One is the case in which both e_A and e_B exceeds $\sqrt{\delta R/2}$, but in this case no Nash equilibrium occurs because candidate A has an incentive to deviate to $e_A = \sqrt{\delta R/2}$ and win the election for sure. In the case of $e_A = e_B < \sqrt{\delta R/2}$, each

candidate has an incentive to slightly increase his promise and win election for sure. Accordingly, in the unique subgame-perfect Nash equilibrium we have $e_A = e_B = \sqrt{\delta R/2}$.

Exercise 4.

To prove the proposition, we need to demonstrate the following two facts.

Fact 1. In equilibrium, $c_{Y1} + c_{Y2} + c_{Y3} = c_{N1} + c_{N2} + c_{N3}$, *that is, the government is made indifferent between the two policies.*
Proof. Suppose that $\sum c_{Yi} > \sum c_{Ni}$ without loss of generality. Then, the government chooses Y and $c_{Yi} > 0$ for some i. This implies that SIG i can increase its payoff, $\pi_{Yi} - c_{Yi}$ by making c_{Yi} a little smaller, keeping $\sum c_{Yi} > \sum c_{Ni}$ to hold. ||

Fact 2. If policy p is chosen in equilibrium, then $\pi_{pi} - c_{pi} \geq \pi_{qi} - c_{qi}$, $p \neq q$, *for every i in equilibrium.*
Proof. Suppose that $\pi_{pi} - c_{pi} < \pi_{qi} - c_{qi}$ for some i. Then from fact 1, SIG i can get better off by making c_{qi} a little larger, inducing the government to change policy from p to q. ||

Now we will prove the proposition. From fact 1, if p is chosen in equilibrium, then

$$\pi_{pi} - c_{pi} \geq \pi_{qi} - c_{qi}, p \neq q,$$

for all i. Summing up each side over $i = 1, 2, 3$, we have

$$\sum \pi_{pi} - \sum c_{pi} \geq \sum \pi_{qi} - \sum c_{qi}. \tag{6}$$

Then from fact 1, $\sum c_{pi} = \sum c_{qi}$, which reduces (6) to $\sum \pi_{pi} \geq \sum \pi_{qi}$. Accordingly, in the case of two policy alternatives, only the one that yields a larger sum of benefits is implemented in equilibrium. ||