Economics of Politics
Lec.4: Perfect Bayesian Nash Equilibrium and Signaling Games

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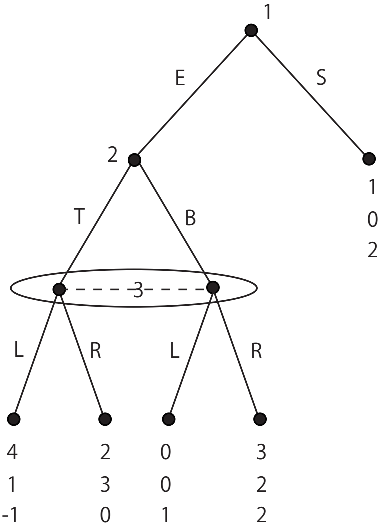
Spring semester, 2012

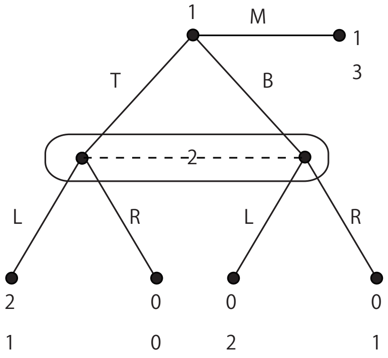
Review: Subgame-Perfect Nash Equilibrium

1. Nash equilibrium: combination of strategies that are mutually best responses.
2. Unnatural beliefs are underlying some Nash equilibria. We like to refine the concept of Nash equilibrium and remove such implausible ones.
3. Subgame-Perfect Nash Equilibrium (SPNE): combination of strategies that constitute a Nash equilibrium in every subgame, including the entire game.
4. In the example on RHS, (S, B, L) and (E, T, R) are Nash equilibria, but only (E, T, R) is a SPNE.



Review: unnatural equilibrium that subgame perfection cannot remove

1. Subgame perfection is not almighty
2. The game on RHS has two Nash equilibria, (T, L) and $(M.R)$, but it seems unnatural for player 1 to believe that player 2 will choose R.
3. However, the whole game is the only subgame of this game, and we cannot refine the equilibria with subgame perfection.
4. We need to introduce a new equilibrium concept that explicitly takes account of beliefs.



Perfect Bayesian Nash Equilibrium (PBNE)

Idea of PBNE

- By making beliefs explicit, we will divide the game that we could not with the concept of subgame perfection, and solve it backward to obtain sequentially rational strategies.

How do you define beliefs?

1. With respect to an information set that contains multiple nodes, we will describe beliefs about which node will be likely to be reached by attaching a probability to each and hence setting a probability distribution over the nodes.
2. Beliefs are assumed to be common across all the players.
3. Beliefs must be consistent with strategies taken.
4. No restriction is placed on beliefs on nodes in the information sets that will not be reached in equilibrium.

Perfect Bayesian Nash Equilibrium (PBNE) 2

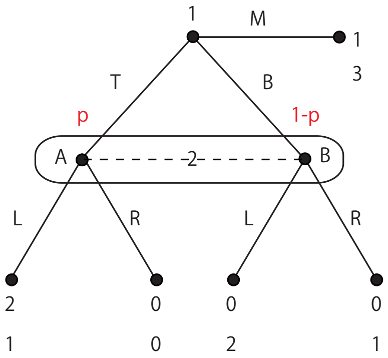
Definition of PBNE

- PBNE is a combination of players' strategies and beliefs.

1. Given beliefs, each player chooses strategies that are sequentially rational, as they are in SPNE.
2. Beliefs are established consistently with players' strategies, in the sense that they do not contradict with Bayes rule.

How to solve a PBNE

1. In the game on RHS. there is only one information set that contains multiple nodes (A and B).
2. A belief is a probability distribution defined over a set of nodes, A and B.
   - If the probability of arriving at A is p, then that of B is $1 - p$.
   - The beliefs in this game can be represented only by specifying a probability , $p \in [0, 1]$.
3. Given such beliefs, player 2's expected payoffs are:
   - $p \times 1 + (1 - p) \times 2 = 2 - p$ if choosing L.
   - $p \times 0 + (1 - p) \times 1 = 1 - p$ if choosing R.
4. Irrespective of p, it is rational for player 2 to choose L whenever she moves.
5. If player 1 anticipates 2's such strategy, it is rational to choose T rather than M.



6. The consistent belief is $p = 1$.
   - Given sequentially-rational strategies, A will be reached for sure.
7. PBNE is $[(T, L), p = 1]$.

Signaling game 1

1. One of the games that address a situation with asymmetric information across players
    - Players are a "sender" and a "receiver." Sender has information that Receiver does not have.
    - Receiver infers the hidden information from Sender's action and decides what to do.
    - Sender uses her own action as a signal to induce Receiver to behave to her advantage.
2. Signaling
    - Examples frequently cited in economics: job hunting (education as a signal), dressing, quality guarantee, advertisement
    - Sending a signal incurs "costs." (cf. attempts to transmit information without paying costs is referred to as "cheap talk.")

Signaling game 2

1. The case of education serving as a signal
   - Receiver (an employer) is uncertain about the ability (productivity) of Sender (a worker).
   - The employer will observe the worker's educational background, from which he infers the worker's ability, and decide whether or not to employ her.
   - The worker tries to improve her education record to get employed, even if higher education needs not only monetary costs like tuition fees but also non-monetary costs such as efforts for studying.
2. The condition for education record to serve as a signal of ability: incentive compatibility
   - For workers with low ability, higher education is too costly to pay if they are employed.
   - For workers with high ability, though being costly, higher education wil pay if they are employed.

Signaling game 3

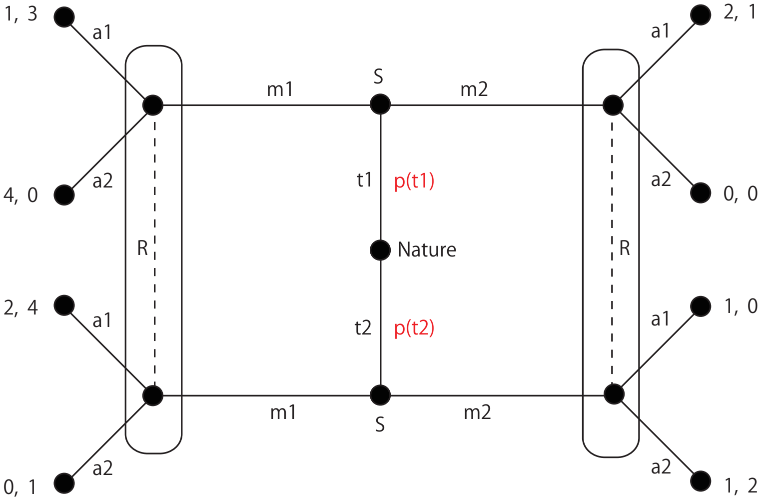
Timing of the game

1. Nature chooses a type of Sender, t_i, out of the set, $T = \{t_1, t_2, \cdots, t_n\}$, at random. Let $p(t_i)$ be the probability of t_i being chosen, which we may call "prior beliefs" and satisfy $\sum_{t_i \in T} p(t_i) = 1$.
2. Sender (S) observes t_i, and choose a message, m_j, out of the set, $M = \{m_1, m_2, \cdots, m_J\}$.
3. Receiver (R) observes m_j, and chooses an action, a_k, out of the set, $K = \{a_1, a_2, \cdots, a_K\}$.
4. S's and R's payoffs are determined like $U_S(t_i, m_j, a_k)$ and $U_R(t_i, m_j, a_K)$, depending on S's type, message, and R's action.

In the case of job signaling:

- S: a worker, R: an employer, $T = \{\text{high ability}, \text{low ability}\}$ $M = \{\text{higher education}, \text{medium education}, \text{lower education}\}$, $A = \{\text{employ}, \text{not employ}\}$

Game tree: the case of T, M, and A having two elements each



Players' strategies and their patterns

1. Sender's strategy
    - Specifies what to do for each of his possible types; that is, designates what message should be sent in response to which type he is.
    - Mathematically, S's strategy is a mapping from T to M, i.e., $m : T \rightarrow M$.
    - Strategy $m(t_i)$ says that S will choose $m(t_i) \in M$ when his type is $t_i \in T$.
2. Receiver's strategy
    - Specifies what to do for each of all possible observations; that is, designates what action should be taken in response to which message she observes.
    - Mathematically, R's strategy is a mapping from M to A, i.e., $a : M \rightarrow A$.
    - Strategy $a(m_j)$ says that R will choose $a(m_j) \in A$ when observing $m_j \in M$.

Patterns in S's strategies and update of R's beliefs

- Three patters in S's strategies
  - Separating strategy: different types will send different messages
  - Pooling strategy: all types will send the same message
  - Hybrid (or Semi-pooling) strategy: not all but some types will send the same message.
- Updating R's prior beliefs into posterior beliefs
  - Prior beliefs may be updated into posterior beliefs with observation of S's messages.
  - R can distinguish S's types from the messages she observes, when S's strategy is separating.
  - R has no information other than prior probability distribution, $p(t_i)$, when S's strategy is pooling.
  - R can sometimes obtain additional information from the messages she observes, when S's strategy is semi-pooling.

A belief system

1. How to introduce beliefs
   - An information set is related exclusively to a message. Different information sets underlie different messages.
   - Different nodes belonging to the same information set are related to different types of Sender.
   - Beliefs are defined as a probability distribution of types conditional on a message observed.
   - Mathematically, beliefs are described by a mapping, $\mu : T \times M \rightarrow [0, 1]$.
   - A belief system $\mu(t_i|m_j)$, which is defined on a space $T \times M$, specifies the probability of Sender's type being t_i when a message, m_j, is observed.
2. Restrictions imposed on a belief system
   - Beliefs must be defined for all possible messages including those that might not be sent in equilibrium.
   - For every m_j, $\sum_{t_i \in T} \mu(t_i|m_j) = 1$.

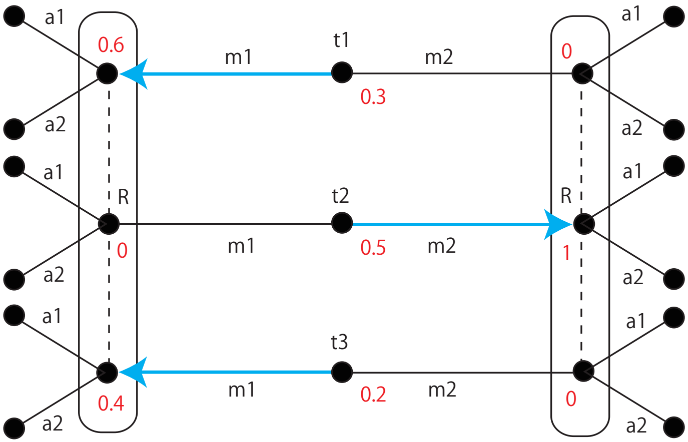
Formation of posterior beliefs consistent with given strategies

1. Consistency with strategies
   - For each message, define a set of S's types that will send it to R, according to his strategy, m.
   - Specifically, let T_j be the set of S's types that choose a common message m_j. It is defined as $T_j \equiv \{t_h | m(t_h) = m_j\}$.
   - For a message with $T_j \neq \emptyset$, in accordance with Bayes rule, posterior beliefs are formed such that
$$\mu(t_i|m_j) = \begin{cases} \dfrac{p(t_i)}{\sum_{t_h \in T_j} p(t_h)} & \text{for } t_i \in T_j \\ \mu(t_i|m_j) = 0 & \text{for } t_i \notin T_j. \end{cases}$$
   - For a message with $T_j = \emptyset$, any belief is allowed as long as it satisfies $\sum_{t_i \in T} \mu(t_i|m_j) = 1$.
   - Beliefs based on massages such that $T_j = \emptyset$ in equilibrium is called "beliefs off the equilibrium path" or "out-of-equilibrium beliefs."
   - No restriction is placed on such beliefs in PBNE. There is a large degree of freedom in beliefs off the equilibrium path.

Beliefs consistent with strategies

Given prior beliefs, $p(t_1) = 0.3,\ p(t_2) = 0.5$, and $p(t_3) = 0.2$, beliefs that are consistent with $m(t_1) = m(t_3) = m_1$ and $m(t_2) = m_2$ are:

- When observing m_1, $\mu(t_1|m_1) = 0.6,\ \mu(t_2|m_1) = 0,\ \mu(t_3|m_1) = 0.4$
- When observing m_2, $\mu(t_1|m_2) = 0,\ \mu(t_2|m_2) = 1,\ \mu(t_3|m_2) = 0$



PBNE of signaling games

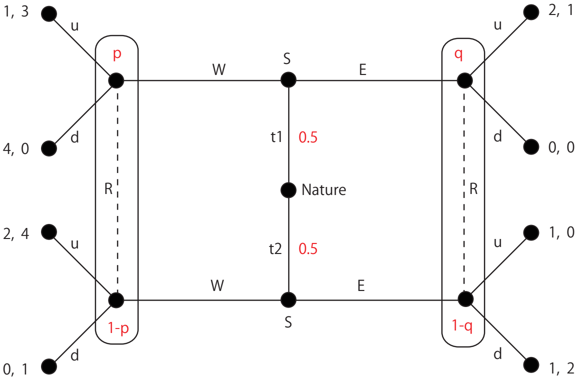
PBNE is a combination of S's strategy m^*, R's strategy a^*, and a belief system μ^*, which satisfies the following conditions.

1. Given μ^*, R who observed a message m_j chooses an action $a^*(m_j)$ that maximizes her expected payoff
$$\sum_{t_i \in T} \mu^*(t_i|m_j) U_R(t_i, m_j, a_k).$$
2. Given μ^*, S whose type is t_i, taking account of R's optimal response $a^*(m_j)$, chooses a message $m_j^*(t_i)$ that maximizes
$$U_S(t_i, m_j, a^*(m_j)).$$
3. The belief system μ^* specifies conditional probabilities of S's type for every possible message as follows, using Bayes rule.
For messages such that $T_j^* \equiv \{t_h | m^*(t_h) = m_j\} \neq \emptyset$,
$$\mu(t_i|m_j) = \frac{p(t_i)}{\sum_{t_h \in T_j} p(t_h)} \text{ for } t_i \in T_j, \quad \mu(t_i|m_j) = 0 \text{ for } t_i \notin T_j.$$
For messages such that $T_j^* = \emptyset$, an arbitrary probability distribution of types is permitted such that $\sum_{t_i \in T} \mu(t_i|m_j) = 1$, as far as it does not change the equilibrium strategies.

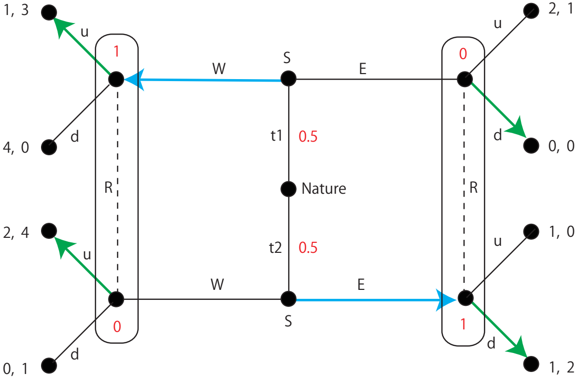
Solve for PBNE

Solve for PBNE of the following game.

- How to solve is to search for separating and pooling equilibria, assuming respective patters of strategies.
- The belief system is (p, q), where p is the probability of player 1's being type 1 when his message is W, and q is the one when his message is E.

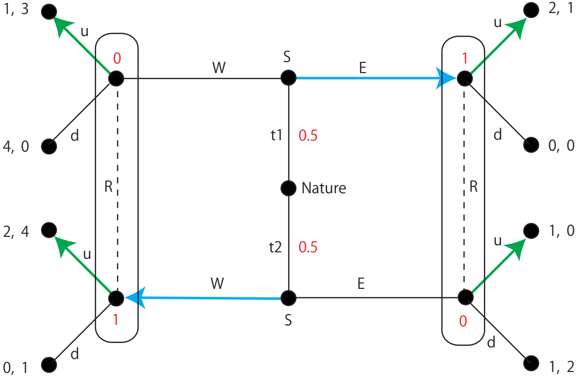


1. Look for a separating equilibrium with $m(t_1) = W$ and $m(t_2) = E$.
   - Consistent beliefs are $p = 1$ and $q = 0$.
   - Given these beliefs, R's optimal strategy is $a(W) = u$ and $a(E) = d$.
   - Taking account of R's optimal strategy, choosing W is optimal for type 1 of S. But for type 2 of S, choosing W is better than choosing E.
   - Thus, there is no such separating equilibrium.



Searching for separating equilibrium 2

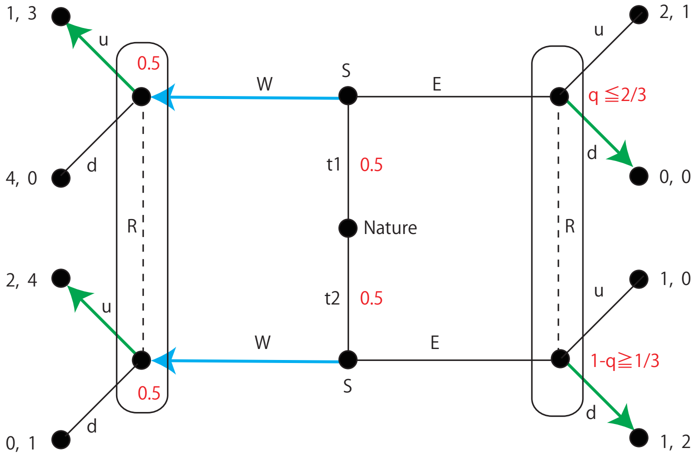
2. Look for a separating equilibrium with $m(t_1) = E$ and $m(t_2) = W$.
    - Consistent beliefs are $p = 0$ and $q = 1$.
    - Taking these beliefs as given, R's optimal strategy is $a(W) = u$ and $a(E) = u$.
    - Anticipating R's best responses, choosing E is optimal for type 1 of S, and choosing W is optimal for type 2 as well.
    - Accordingly, such strategies and beliefs constitutes a PBNE.



Searching for pooling equilibrium 1

1. Look for a pooling equilibrium with $m(t_1) = m(t_2) = W$.
    - In consistent beliefs, $p = 0.5$ and q is arbitrary (of course, $0 \leq q \leq 1$).
    - Given these beliefs, R's optimal response is $a(W) = u$ when she observes W. When observing E, it is $a(E) = u$ if $q \geq 2/3$, and $a(E) = d$ if $q \leq 2/3$.
    - Suppose $q \geq 2/3$. Then, taking account of R's best responses, type 1 of S will choose E, not W. Hence, there is no such an equilibirum.
    - Suppose $q \leq 2/3$. Then, taking account of R's best responses, it is optimal for type 1 of S to choose W, and so it is for type 2 to choose W. Hence, such a pooling equilibrium exists if and only if $q \leq 2/3$.
    - Since PBNE places no restriction on the out-of-equilibrium beliefs, any $q \leq 2/3$ is permitted.

Searching for pooling equilibrium 2



Searching for pooling equilibrium 3

2. Look for a pooling equilibrium with $m(t_1) = m(t_2) = E$.
    - In consistent beliefs, p is arbitrary and $q = 0.5$.
    - Given these beliefs, R's best responses are $a(W) = u$ and $a(E) = d$.
    - Incorporating R's best responses, choosing W, not E, is optimal for type 1 of S.
    - Thus, there is no such pooling equilibrium.